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Vicious walkers and Young tableaux I: without walls

Anthony J Guttmann†, Aleksander L Owczarek† and Xavier G Viennot‡

† Department of Mathematics and Statistics, The University of Melbourne, Parkville, Victoria 3052, Australia

‡ LaBRI, Université Bordeaux 1, 351 cours de la Libération, 33405 Talence Cedex, France

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Abstract. We rederive previously known results for the number of star and watermelon configurations by showing that these follow immediately from standard results in the theory of Young tableaux and integer partitions. In this way we provide a proof of a result, previously only conjectured, for the total number of stars.

1. Introduction

In an earlier paper [1] the problem of vicious random walkers on a D-dimensional directed lattice was considered. 'Vicious walkers' describes the situation in which two or more walkers arriving at the same lattice site annihilate one another. Accordingly, the only configurations allowed are those in which contacts are forbidden. Alternatively expressed, we consider mutually self-avoiding networks of directed lattice walks, which also model directed polymer networks. The connection of these vicious walker problems to the six-vertex model in statistical mechanics is also discussed.

The problem, together with a number of physical applications, was introduced by Fisher [2]. The general model is one of p random walkers on a D-dimensional lattice who at regular time intervals simultaneously take one step with equal probability in the direction of one of the allowed lattice vectors.

The two standard topologies of interest are that of a *star* and a *watermelon*. Consider a directed square lattice, rotated 45°, so that the unit vectors on the lattice are $(\boldsymbol{i}+\boldsymbol{j})/\sqrt{(}2)$ and $(\boldsymbol{i}-\boldsymbol{j})/\sqrt{(}2)$. Both configurations consist of p chains of length n which start at $(0,0), (0,2), (0,4), \ldots, (0,2p-2)$. The watermelon configurations end at $(n,k), (n,2+k), (n,4+k), \ldots, (n,k+2p-2)$. For stars, the endpoints of the chains all have x-coordinate equal to n, but the y-coordinates are unconstrained, apart from the ordering imposed by the non-crossing condition. Thus if the endpoints are $(n,e_1), (n,e_2), (n,e_3), \ldots, (n,e_p)$, then $e_1 < e_2 < e_3 < \cdots < e_p \leqslant 2p-2+n$.

In [1] recurrence relations and the corresponding differential equations for stars and watermelons on the directed square lattice were obtained. In the case of watermelons, a determinental form was evaluated by standard techniques applied to the determinant. In the case of stars, the results obtained were conjectural, being equivalent to an earlier conjecture [3].

In this paper we show how a number of 'standard' results in the theory of Young tableaux and partitions lead to a much more intuitive derivation of the above results. Furthermore, the conjectured results are proved.

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As well as providing an alternative and more intuitive derivation of earlier results and proving conjectures, this paper has a pedagogical purpose. We wish to introduce the properties of Young tableaux, previously the domain of combinatorialists, to this domain of physics. The power of these techniques will be further demonstrated in a subsequent paper, in which we deal with the more difficult problems of stars and watermelons in the presence of an impenetrable wall, and then sandwiched between two walls.

2. Determinants and self-avoiding networks of walks

Let $\mathbb{L}$ be an arbitrary set and $\mathbb{K}$ a commutative ring. Usually $\mathbb{L}$ will be chosen to be the square lattice $\mathbb{Z} \times \mathbb{Z}$, and $\mathbb{K}$ will usually be the ring $\mathbb{Z}[\alpha, \beta, \ldots]$ of polynomials in variables $\alpha, \beta, \ldots$ and coefficients in $\mathbb{Z}$. A *valuation* (also called a *weight*) is any function $v : \mathbb{L} \times \mathbb{L} \to \mathbb{K}$. A *walk* w (or *path*) is any sequence $w = (s_0, \ldots, s_n)$. We say that w *starts* at s_0 and *ends* at s_n. Each pair (s_i, s_{i+1}) is an *elementary step* of w. The *weight* $v(w)$ of a walk is the product of the weights of its elementary steps, that is $v(w) = v(s_0, s_1) \ldots v(s_{n-1}, s_n)$.

Let $(A_1, \ldots, A_k)$ and $(B_1, \ldots, B_k)$ be two sequences of points of $\mathbb{L}$. We assume the following finiteness condition.

(F) For any i, j, such that $1 \leqslant i, j \leqslant k$, the set $W_{i,j}$ of walks starting at A_i and ending at B_j, with non-zero weight, is finite.

This condition implies that the paths of $W_{i,j}$ are self-avoiding. We can define the sum $a_{i,j} = \sum_{w \in W_{i,j}} v(w)$. If $v(s, t) = 0$ or 1, then $a_{i,j}$ is the number of paths of $W_{i,j}$. The *crossing condition* is the following.

(C) For any i, j, i', j', such that $1 \leqslant i < i' \leqslant k$, $1 \leqslant j < j' \leqslant k$, $w \in W_{i,j'}$, $\eta \in W_{i',j}$, such that $v(w) \neq 0$ and $v(\eta) \neq 0$, then $w \bigcap \eta \neq 0$. (That is to say, the two paths w and v *intersect* or have a common vertex.)

We can now state the following theorems.

Theorem 1 (Gessel and Viennot [4]). Let $\mathbb{L}, v(A_1, \ldots, A_k), (B_1, \ldots, B_k)$ satisfy both the finiteness and crossing conditions. Then the determinant of the matrix $(a_{i,j})_{1 \leqslant i,j \leqslant k}$ is the sum of the weights of all configurations of walks $\Omega = (w_1, \ldots, w_k)$ satisfying the following two conditions.

(i) The walks of Ω are pairwise disjoint, and

(ii) each walk w_i goes from A_i to B_i.

In other words $\det(a_{i,j})_{1 \leqslant i,j \leqslant k} = \sum_{\Omega=(w_1,\ldots,w_k),(i),(ii)} v(w_1) \ldots v(w_k)$.

This theorem was in fact anticipated by Lindström [5] in the context of matroids. The application of this basic property to combinatorics was developed by Gessel and Viennot [4], and subsequently widely used. In the combinatorial literature it is called the Gessel–Viennot methodology.

A particular case is the *vicious walker* problem. In that case Fisher [2] showed the appearance of the above determinant. In the same context, a probabilistic version of the determinant was given by Karlin and McGregor [6].

Here we sketch the proof of the above theorem. By definition, the determinant of the matrix $(a_{i,j})$ is

$$\det(a_{i,j})_{1 \leqslant i,j \leqslant k} = \sum_{(\sigma,\Omega)} (-1)^{\mathrm{Inv}(\sigma)} v(w_1) \ldots v(w_k)$$

where the summation is over all permutations σ of the symmetric group σ_n. The configuration $\Omega = (w_1, \ldots, w_k)$ is such that the paths w_i go from A_i to $B_{\sigma(i)}$ and $\mathrm{Inv}(\sigma)$ denotes the number of *inversions* of the permutation σ.

In this summation, many terms have the same weight with opposite signs. Most of them cancel pairwise. It can be shown that the remaining terms correspond to configurations of the pairwise disjoint walks. Moreover the crossing condition implies that the remaining terms correspond to the identity permutation. In fact the theorem holds in the following more general form.

Theorem 2. If $\mathbb{L}$, v, $(A_1, \ldots, A_k)$, $(B_1, \ldots, B_k)$ satisfy the finiteness condition, then

$$\det(a_{i,j})_{1\leqslant i,j\leqslant k} = \sum_{(\sigma,\Omega)} (-1)^{\mathrm{Inv}(\sigma)} v(w_1)\ldots v(w_k)$$

with the summation restricted to $\sigma \in \sigma_n$, $\Omega = (w_1, \ldots, w_k)$ and is over all pairs (σ, Ω) satisfying (i) and (ii). Furthermore, each walk w_i goes from A_i to $B_{\sigma(i)}$.

We illustrate the theorem by an example, that of binomial determinants [4].

Let $\mathbb{L} = \mathbb{Z} \times \mathbb{Z}$, and let v be the valuation defined by the following: $v(s, t) = 1$ iff t is a neighbour of s located just to the east or north of s and the points s and t are inside the octant $0 \leqslant y \leqslant x$.

For any point $A = (i, 0)$, $i \geqslant 0$, on the vertical axis, and $B = (j, j)$, $j \geqslant 0$, on the diagonal, the set $W(A, B)$ of walks going from A to B with non-zero weight $v(W)$ is nothing but the set of paths going from A to B with elementary steps north or east.

Let P be the 'enlarged' Pascal triangle, $P = \binom{i}{j}_{0\leqslant i,j}$ where the binomial coefficient, $\binom{i}{j}$ is zero when $j > i$. A *binomial determinant* is *any minor of the infinite matrix* P. Let $0 \leqslant a_1 < \cdots < a_k$ and $0 \leqslant b_1 < \cdots < b_k$ be two strictly increasing sequences of non-negative integers. The minor with row indices $(a_1, \ldots, a_k)$ and column indices $(b_1, \ldots, b_k)$ is denoted by the following

$$\begin{pmatrix} a_1, \ldots, a_k \\ b_1, \ldots, b_k \end{pmatrix} = \det \begin{pmatrix} a_i \\ b_j \end{pmatrix}_{1\leqslant i,j\leqslant k}.$$

The notation is consistent with the notation of the binomial coefficient when $k = 1$. For $1 \leqslant i \leqslant k$, let $A_i = (a_i, 0)$ and $B_j = (b_j, b_j)$. We then have

$$\sum_{w\in W(A_i,B_j)} v(w) = \begin{pmatrix} a_i \\ b_j \end{pmatrix}$$

that is the number of walks going from A_i to B_j and having elementary steps east and north.

The crossing condition is clearly satisfied. An immediate corollary is that any binomial determinant is non-negative.

In the case of vicious walkers on a line as in [2], the crossing condition is satisfied and Fisher's determinant may be deduced from theorem 1. From the same theorem, we can also deduce the determinants of [7, 8].

In the paper of Forrester [7], which treated vicious walkers on a cylinder, the crossing condition (C) is not satisfied. Nevertheless, for an odd number of walkers, we can still apply the Gessel–Viennot methodology as if the crossing condition were satisfied. This is because the permutations appearing in the summation in theorem 2 are only circular permutations, and for n odd, such permutations have an even number of inversions. Forrester's proposition then follows from theorem 2.

In [4] two important cases are given in which the binomial determinant can be calculated as a ratio of two products. This is possible when the points $A_1, \ldots, A_k$ are consecutive, or when the points $B_1, \ldots, B_k$ are consecutive (see also section 4 below).

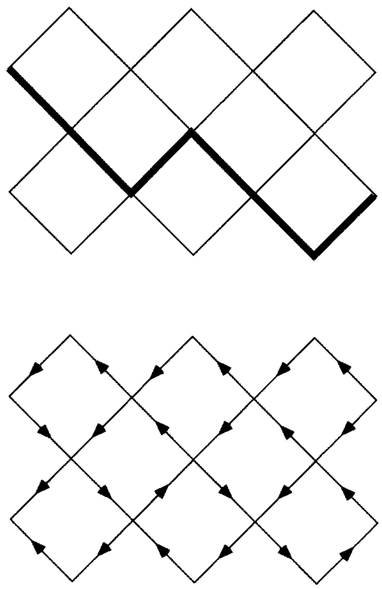


Figure 1. A simple path configuration on the top lattice is mapped onto the bottom arrow configuration by orienting the path one way and all other edges of the lattice the other way.

3. Vertex models and networks of directed walkers

For the moment let us examine the standard case where the valuations are as described above, and so the Gessel–Viennot determinant evaluates the numbers of configurations of walks on the lattice. We first place arrows on the walks such that they point in a direction that has a component in the positive x-direction. On the rest of the lattice, that is on the empty bonds, place arrows the opposite way. If we examine each vertex we notice that there are two arrows into, and two arrows out of, each vertex: see figure 1. This is the so-called *ice rule* [9–11] and maps each configuration of walks onto a configuration of the six-vertex model of lattice statistical mechanics. Archetypical special cases were first solved—'square ice' [10, 12]; Slater KDP [13]; Rys F model [14]—before the more general models were considered: see [15] (symmetric six-vertex model: where the Boltzmann weights associated with vertex configurations obtained from one another by reversing each edge arrow are equal) and [16] (asymmetric six-vertex model where all six associated Boltzmann weights can be different). General discussions of the six-vertex model and its history can be found in [17, 11].

To complete the mapping one must consider the resulting Boltzmann weights of the vertex model: see figure 2. The six weights of the vertex model $\{w_1, w_2, w_3, w_4, w_5, w_6\}$ are given as $\{1, 0, 1, 1, 1, 1\}$. We note that one weight is zero, corresponding to the vicious-walker constraint. For convenience we shall denote the vertex model with these weights as the vicious-walker vertex model. Our resultant vertex model is a special case of the it modified KDP model of Wu [18, 19], which is itself a specialization of the full asymmetric (all weights possibly different) six-vertex model [16]. (Full details of the solution of the most general case announced in [16] have never been published, although many subcases have appeared in the literature!) (This special case can be compared with some others: 'square ice' [10, 12] has all weights equal, the Slater KDP [13] has weights $\{1, 1, w, w, w, w\}$, the Rys F model [14] has weights $\{w, w, w, w, 1, 1\}$, while the symmetric six-vertex model [15] has weights $\{a, a, b, b, c, c\}$.) If we were to change the valuations so as to add a variable that counted the total number of bonds in the walks we would have a full realization of

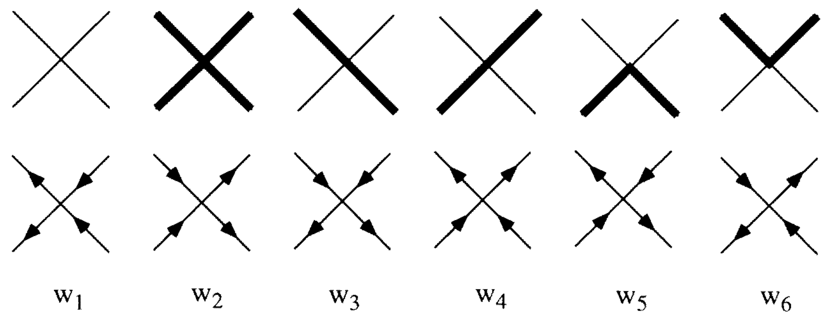


Figure 2. At the top are the six possible path configurations at a vertex of the lattice. Below each of these is the associated arrow configuration of the six-vertex model, while below that is the six weights we associate with each of those configurations. Note that different authors use different numbering.

the modified KDP model. The standard quantity for calculation in a statistical mechanics model is the partition function, defined in this case as

$$Z^{6V} = \sum_{\text{arrow configurations}} \prod_{\text{vertices of the lattice}} (\text{Boltzmann weights}). \tag{3.1}$$

Re-interpreted in terms of our path problem, this implies that the vertex partition function calculates the sum of weighted paths over *all* possible configurations of *any* number of walks. Note that in the calculation of the partition function each weight vertex on the lattice can be trivially multiplied by a common factor f without effecting the difficulty of the calculation (giving simply a factor $f^{(\# \text{ vertices})}$).

This mapping of vertex configurations to bond or walk configurations (sometimes considered as domain wall and crystal surface configurations [20, 21, 2, 22]) has been well known for some time [18]. Note that the traditional setting of the vertex model lattice has not been at 45° as here, though such cases have been considered [21, 23]. Of course, for thermodynamic vertex model quantities, such as the free energy, the orientation of the lattice is unimportant.

We must however be careful in identifying the problem of counting walkers starting from a fixed set of positions to a fixed set of ending positions with the thermodynamic calculations of vertex models which flow from the partition function described above. To be precise, we need to note the following. By fixing the numbers of walks we are considering an invariant subspace of the transfer matrix described by the number of 'forward' arrows. Also, the transfer matrix is most naturally defined on a finite strip: if we want to consider walks without walls we must choose our strip wide enough to ensure that walk configurations never touch the strip edge (or, with periodic boundary conditions, that the top walk does not approach the bottom walk). Most importantly, when we calculate the number of walks with fixed starting and ending positions we are evaluating the appropriate matrix element of the transfer matrix raised to the length of the walks n. That is, we find

$$\langle \phi^f | T^n | \phi^i \rangle \tag{3.2}$$

where T is the transfer matrix and $\langle \phi_f |$ and $| \phi_i \rangle$ are vectors describing the initial and final states (the positions of the walkers). To evaluate this one may spectrally decompose the transfer matrix T into

$$T^n = \sum_j |\psi_j\rangle \lambda_j^n \langle \psi_j | \tag{3.3}$$

where $|\psi_j\rangle$ and λ_j are the normalized eigenvectors and eigenvalues respectively of T. This is akin to calculating the correlation functions of the vertex model rather than the free energy (which is simply related to the maximum eigenvalue), if it were not for the subspace restriction! On the other hand summing over both sets of endpoints gives us a reduced partition function for that sector (which can be found from the eigenvalues of that sector alone). Hence calculating weighted path sums is somewhere between calculating correlation functions and the partition function in difficulty.

Some further technical points to note are that, first, the transfer matrix that is usually defined is an edge-to-edge transfer matrix acting on a vector space describing the 'states' of rows of edges. Second, in the 45° direction, either, two different transfer matrices are needed to account for the bipartite nature of the vertices, or one needs to consider two independent interlocking lattices: we refer the interested reader to [24] for more details.

We also make some observations about the vertex model onto which our vicious walker problem maps. With the weights as given above the model satisfies the so-called *free-fermion* condition [25] which implies that the vertex model [26] can also be solved by the method of Pfaffians [27]. Here the vertex model on the square lattice is mapped [18, 19] onto a dimer problem [28, 27] on the honeycomb lattice [29–31]. For the six-vertex model the free-fermion condition states

$$w_1w_2 + w_3w_4 = w_5w_6. \tag{3.4}$$

The origin of the name 'free fermion' lies in a field-theoretic (S-matrix) approach to the problem [29, 32]. Also, in the more general eight-vertex setting the conditions $w_1w_2 = w_7w_8$ and $w_3w_4 = w_5w_6$ [32, 17] allow the vertex model to be mapped to a nearest-neighbour Ising model on the triangular lattice, the transfer matrix of which can be written in terms of fermionic operators and hence solved by the Schultz, Mattis and Lieb approach [33]. (The six-vertex model has $w_7 = w_8 = 0$ and so our vicious-walker model satisfies those conditions, though the resultant Ising model has some infinite coupling strengths.)

Importantly in our context, the free-fermion cases of the six-vertex model admit the calculation and analysis of correlation functions [34, 35], which we mentioned above is similar to the calculation of the walk configuration numbers (in that both need *normalized eigenvectors* as well as eigenvalues to be calculated). This calculation is possible because the eigenvectors of the transfer matrix used in the calculation are given in terms of the Bethe ansatz, which is a superposition of plane wave states. Moreover, the wavenumbers of these states are usually given by a set of nonlinear coupled equations called the Bethe ansatz equations. When the free-fermion condition holds these equations decouple and an explicit solution can be found in terms of a set of wavenumbers, each of which is a valid solution of the one forward arrow sector equation (one-walker problem). In fact the n-walker eigenfunction becomes simply a sum of a product of the one-walker eigenfunctions. This sum of products is the evaluation of a determinant [34]. These determinant forms parallel the general path results of Gessel–Viennot [4] for a different type of calculation. A more explicit explanation of the relationship between the transfer matrix approach to vertex models and the determinantal formulae for lattice paths can be found in [24].

To end this section we note that the vertex-walk model mapping has recently allowed calculation and analysis of other interacting cases using the Bethe ansatz solution of the six-vertex model [36, 37]. Also, a generalized case of the vicious-walker vertex model, still with $w_2 = 0$ but falling outside the free-fermion condition generally, known as the five-vertex model, has recently been solved as a model of interacting dimers [38].

4. Stars with fixed endpoints

We now revert to the vicious-walker problem and show the application of theorem 1 to the situation of stars. As shown in [1], the first step in these lattice path problems is to write down a $p \times p$ determinant whose (i, j)th element is the binomial coefficient giving the number of paths from the ith starting point to the jth endpoint. That is to say, the number of stars is given by

$$\det_{1\leqslant i,j\leqslant p}(|P(A_j \to E_i)|) \tag{4.1}$$

where $P(A \to E)$ denotes the set of all lattice paths from A to E. Each path number $|P(A_j \to E_i)|$ is a binomial, so that (4.1) equals

$$\det_{1\leqslant i,j\leqslant p}\left(\binom{n}{\frac{n+e_i}{2} - j + 1}\right).$$

This result was first given explicitly in [39, corollary 2] (see also [40, theorem 1.2]) but, as noted above, is implicit in earlier work [5, 6].

The determinant can be expressed as a product formula, as shown in appendix C of [1], explicitly

$$2^{-\binom{p}{2}}\prod_{i=1}^{p}\frac{(n-i+p+1)!}{\left(\frac{n+e_i}{2}\right)\left(\frac{n-e_i}{2}+p-1\right)!}\prod_{1\leqslant i<j\leqslant p}(e_j - e_i). \tag{4.2}$$

In [1] the result was obtained by first principles, that is to say, by a sequence of row and column operations applied to the determinant.

These operations can be greatly simplified by using a lemma [41, lemma 2.2], which has in turn been shown to be deriveable from Dodgson's formula [42].

Lemma 1. Let $X_1, X_2, \ldots, X_p, A_2, \ldots, A_p, B_2, \ldots, B_p$ be any entries in the determinant. Then

$$\begin{aligned}\det_{1\leqslant i,j\leqslant p}&((X_i + A_p)\ldots(X_i + A_{j+1})(X_i + B_j)\ldots(X_i + B_2))\\ &= \left(\prod_{1\leqslant i<j\leqslant p}(X_i - X_j)\right)\left(\prod_{2\leqslant i\leqslant j\leqslant p}(B_i - A_j)\right).\end{aligned} \tag{4.3}$$

An alternative, and easier proof of (4.2) is then obtained by removing as many common factors from the determinant as possible, giving

$$\begin{aligned}(-1)^{\binom{p}{2}}&\frac{(n!)^p}{\prod_{i=1}^{p}\left(\frac{n+e_i}{2}\right)\left(\frac{n-e_i}{2}+p-1\right)!}\det_{1\leqslant i,j\leqslant p}\left(\left(\frac{-n+e_i}{2}-p+1\right)\ldots\left(\frac{-n+e_i}{2}-j\right)\right.\\ &\left.\times\left(\frac{n+e_i}{2}-j+2\right)\ldots\left(\frac{n+e_i}{2}\right)\right).\end{aligned}$$

Now applying the above lemma with $X_i = e_i/2$, $A_i = -n/2 - i + 1$, $B_i = n/2 - i + 2$, a little simplification yields (4.2).

4.1. Total number of stars

To obtain the *total* number of stars (no fixed endpoints) one must sum (4.2) over all possible endpoints. A *conjectured* product formula is given in [1]. This is

$$\prod_{1\leqslant i\leqslant j\leqslant n}\frac{p+i+j-1}{i+j-1}. \tag{4.4}$$

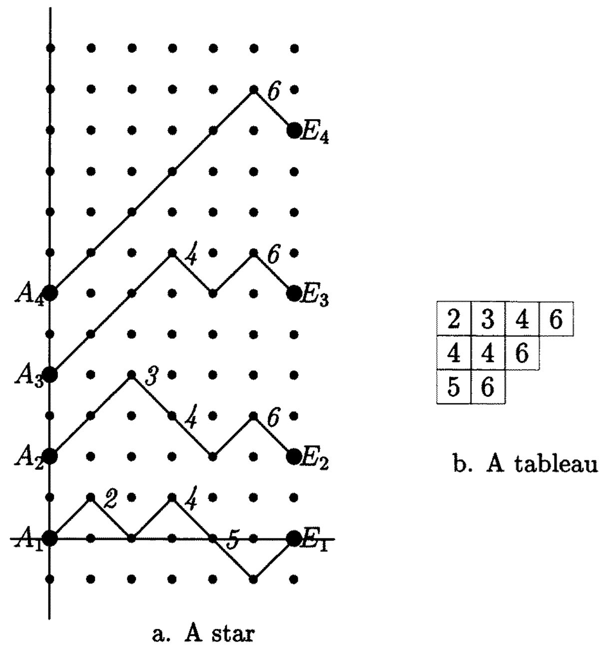


Figure 3. (a) A typical star configuration and (b) associated tableau.

As also pointed out in [1], this is equivalent to another conjecture in [3]. We will show below that (4.4) follows immediately from standard results in the theory of tableaux, thereby proving the above result and the equivalent conjecture.

5. Correspondence between stars and Young tableaux

There is a standard bijection between stars with p branches of length n and semi-standard Young tableaux with entries $\{1, 2, \ldots, n\}$ having at most p columns.

A typical star configuration is shown in figure 3. To each path going from A_i to E_i ($1 \leqslant i \leqslant p$) we associate the increasing sequence of integers formed by the x-coordinate of the endpoint of its south-west steps. Placing these integers in columns (increasing from top to bottom) and writing the columns from left to right (corresponding to paths $1, 2, \ldots, p$), we obtain the tableau shown. Note that the ith branch runs from $A_i = (0, 2i - 2)$ to $E_i = (n, e_i)$, $i = 1, 2, \ldots, p$. In figure 3(a), $p = 4$, $n = 6$, $e_1 = 0$, $e_2 = 2$, $e_3 = 6$, $e_4 = 10$.

The *shape* of the tableau is determined by the endpoints of the star. The shape is $\boldsymbol{\lambda} = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$ where λ_i is the number of elements in the ith row, and the entries are in $\{1, 2, \ldots, n\}$.

Theorem 3. There is a one-to-one correspondence (a bijection) between non-crossing stars with p branches of length n and semistandard Young tableaux with entries in $\{1, 2, \ldots, n\}$ having at most p columns.

A straightforward proof relies on formalizing the observation that each star gives a unique tableau, and that each tableau gives a unique star.

The shape of the tableau is determined by the endpoints $E_1, E_2, \ldots, E_p$. A moment's thought suffices to establish that watermelons are in bijection with *rectangular* tableaux.

7	5	3	2
6	4	2	1
3	1		
1			

a. Hook lengths

0	1	2	3
−1	0	1	2
−2	−1		
−3			

b. Contents

Figure 4. Examples of (a) the hook lengths and (b) the contents of a tableau.

Having shown the correspondence between stars, watermelons and tableaux, an alternative, and more powerful method of the proof of these and other results relies on established results in the theory of tableaux. Thus, for example, to count the number of stars with fixed endpoints, we need only count the total number of tableau of a given shape—corresponding to the star. The total number of stars is obtained by counting the total number of tableaux. These observations are formalized in the following sections.

We first require the definition of two additional concepts, that of the *content* and *hook length* of a tableau.

For each cell the *hook length* is defined as 1 plus the number of cells below and to the right of the given cell, as shown in figure 4(a). The *content* of cell (i, j) is simply $j - i$, thus the contents are obtained by labelling the main diagonal cells by 0, the diagonal above the main diagonal has all cells labelled 1, and the diagonal below the main diagonal has all cells labelled -1 (see figure 4(b)). Diagonals two removed from the main diagonal have all their cells labelled ± 2 according to whether they are above or below the main diagonal.

Clearly, these are properties merely of the tableau's *shape.* A standard result gives the number of such tableaux. It was first proved by Stanley [43], who gave an inductive proof. As a result in the theory of non-crossing paths it appears in [39].

The required result is that the number of tableau of a given shape can be written as a quotient of the form $\frac{\text{product}}{\text{product}}$, where the denominator is a product over hook lengths $\prod_x h_x$, and the numerator is $\prod_x (n + c_x)$. The product is taken over all cells x of the tableau. h_x denotes the hook length of cell x, and c_x denotes the content of cell x. Recall that n is just the length of each path.

In a given tableau, the column lengths are given by $\frac{n-e_1}{2}, \frac{n-e_2}{2}, \ldots, \frac{n-e_p}{2}$. The shape is just the vector of row lengths, which can readily be extracted from the column lengths. From the shape, the content readily follows. In this way, the product $\prod_x \frac{n+c_x}{h_x}$ can be converted to (4.2).

An alternative, and simple proof based on the properties of Schur functions can also be given. As this proof requires the more powerful machinery of the theory of symmetric functions, it will be given in a subsequent paper, in which we consider the problem of enumerating stars and watermelons in the presence of an impenetrable wall.

5.1. Total number of stars

The conjectured result for the *total* number of stars (4.4) can readily be proved by appealing to a well known result in the theory of tableau. Thus, we have the following theorem.

Theorem 4. The number of stars of length n with p branches equals

$$\prod_{1\leqslant i\leqslant j\leqslant n} \frac{p+i+j-1}{i+j-1}. \tag{5.1}$$

Using the correspondence between stars and tableaux described in theorem 3, we see that we wish to count the total number of tableaux with entries at most n having at most p columns.

The result of this enumeration problem (actually the more difficult q-analogue problem) was known for many years under the name *Bender–Knuth conjecture*, and was first proved by Gordon around 1970 (but appeared only much later as [44]). Taking the limit $q \to 1$ immediately gives (5.1).

The proof of the *Bender–Knuth conjecture* will also be given in a subsequent paper, once the relevant machinery is developed.

5.2. Watermelons with given deviation

Let $\bar{W}^p_{m,k}$ be the number of watermelons having length m, with p hairs, and deviation $k > 0$. (The case $k < 0$ is obtained by reflection from the case $k > 0$.) The starting points are $(0, 0), (0, 2), \ldots, (0, 2p-2)$, and the ending points are $(m, k), (m, k+2), \ldots, (m, k+2p-2)$. A formula for $\bar{W}^p_{m,k}$ written as $\frac{\text{product}}{\text{product}}$ was derived in [1], as a special case of (4.2). We show how it can be obtained combinatorially as follows, following the prescription of the previous section.

From theorem 3 it follows that such a watermelon configuration corresponds to a rectangular tableau with p columns and $n = \frac{m-k}{2}$ rows. The hook length and content of the tableau is shown below.

$n+p-1$	$\cdots$	$n+2$	$n+1$	n
$n+p-2$	$\cdots$	$n+1$	n	$n-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$p+1$	$\cdots$	4	3	2
p	$\cdots$	3	2	1

The above table gives the hook lengths h_x, while the contents c_x are given by

0	1	2	$\cdots$	$p-1$
-1	0	1	$\cdots$	$p-2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2-n$	$3-n$	$4-n$	$\cdots$	$p-n+1$
$1-n$	$2-n$	$3-n$	$\cdots$	$p-n$

Then

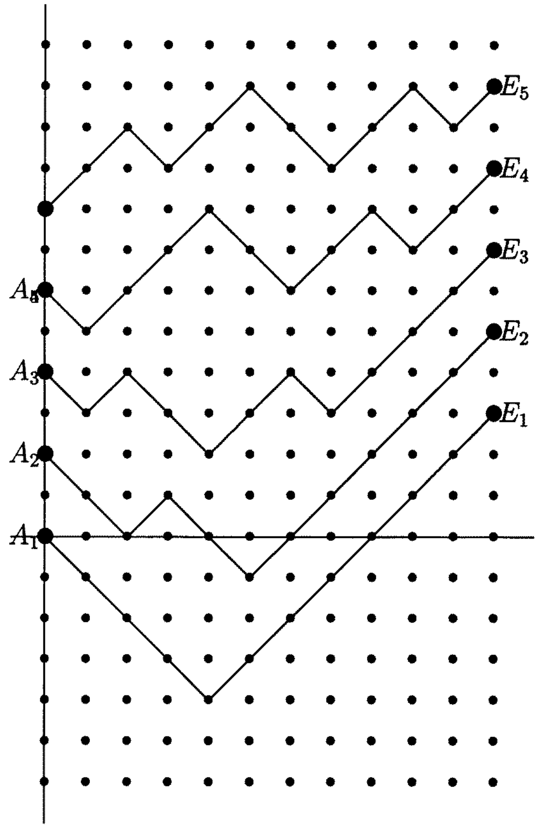
$$\bar{W}^p_{m,k} = \frac{\prod_x (m + c_x)}{\prod_x h_x} = \prod_{i=1}^{p} \prod_{j=1}^{n} \frac{m+i-j}{i+j-1} \tag{5.2}$$

where the denominator is given by the product of the hook lengths and the numerator by the product of the contents c_x shifted by m for all cells x of the tableau. For small deviations ($k = 0$ and $m = 2n$) and ($k = 1$ and $m = 2n+1$), this product simplifies to

$$\bar{W}^p_{2n,0} = \prod_{1 \leqslant i,j \leqslant n} \frac{p+i+j-1}{i+j-1} \tag{5.3}$$

$$\bar{W}^p_{2n+1,1} = \prod_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n+1} \frac{p+i+j-1}{i+j-1}. \tag{5.4}$$

At first sight the equality of (5.2) and (5.3), (5.4) seems unlikely, as there are np terms in the numerator and denominator of (5.2), compared with n^2 terms in the numerator and



a. Watermelon

b. Plane partition

4	3	3	2	2	1
4	2	2	1		
3	2	2	1		
1	1				

Figure 5. A watermelon configuration and associated plane partition.

denominator of (5.3) and n^2+n terms in the numerator and denominator of (5.4). However, it is readily seen that if $n > p$, the terms in the numerator of (5.3) with $i < n-p+1$ exactly cancel the denominator terms with $i > p$. If $n < p$, the terms corresponding to the first $p-n$ columns in the tableau contributing to both the numerator and denominator cancel. Similar considerations establish the equality of (5.2) and (5.4) for the case $m = 2n+1$.

5.3. Watermelons and plane partitions

It is instructive to consider another bijection, that between watermelons and plane partitions. In this case, watermelons enumerated by $\bar{W}^p_{m,k}$ are in bijection with classical *plane partitions* embedded in $r \times n$ rectangles, with $n = (m+k)/2$, $r = (m-k)/2$ and with parts $\leqslant p$.

A *plane partition* is a tableau $a_{i,j}$ of non-negative integers (called *parts*) such that these integers are weakly decreasing in both rows and columns (from both left to right and from top to bottom). That is to say, $a_{i,j} \geqslant a_{i,j+1} \geqslant 0$, and $a_{i,j} \geqslant a_{i+1,j} \geqslant 0$. The plane partition of figure 5 has a *shape* which is embedded in a 4×7 rectangle. It is constructed by collapsing all the paths, so that the path starting at $A_i = (0, 2i-2)$ has the y-coordinate of every point reduced by $2i-2$.

Refer to figure 5. By vertically translating each path downwards so that the starting and ending vertex coincide, we get all paths in an $n \times r$ rectangle (in this example $r = 4$, $n = 7$) and all starting in the top, left corner of the rectangle and all finishing in the bottom, right corner of the rectangle.

The bijection is obtained by first rotating the partition 45° clockwise (so that its orientation on the page is the normal one) and then labelling each cell of the $r \times n$ rectangle by the number of paths passing above the north-east corner of that cell. The plane partition is then obtained by deleting all the 0 entries. The largest *part* or entry is the number of paths different from the path formed by r south steps followed by n east steps. Thus the biggest part is $\leqslant p$. This is a one-to-one correspondence.

If we now put a stack of $a_{i,j}$ cubes on each cell of the $r \times n$ rectangle, we obtain the three-dimensional *solid representation* of the plane partition (or three-dimensional Ferrers diagram), in which the diagram is embedded in an $r \times n \times p$ box.

Such solid diagrams, or plane partitions, have been enumerated in [45] as

$$P_{r,n,p} = \prod_{i=1}^{r} \prod_{j=1}^{n} \prod_{l=1}^{p} \frac{i+j+l-1}{i+j+l-2}. \tag{5.5}$$

This formula is equivalent to

$$P_{r,n,p} = \prod_{i=1}^{r} \prod_{j=1}^{n} \frac{p+i+j-1}{i+j-1} \tag{5.6}$$

which is just the formula above for the number of watermelons.

6. Discussion and conclusion

In this paper we have shown that by appealing to standard results in the theory of Young tableaux and plane partitions, results obtained previously by direct algebraic means can be obtained more directly, and arguably more transparently. By this means we prove a result that was previously only conjectured.

In a subsequent paper, we shall show how these same techniques can fairly readily provide exact results for the more difficult lattice path problem in the presence of a wall.

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